The Energy Density in the Casimir Effect

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Abstract

We compute the expectations of the squares of the electric and magnetic fields in the vacuum region outside a half-space filled with a uniform dispersive dielectric. We find a positive energy density of the electromagnetic field which diverges at the interface despite the inclusion of dispersion in the calculation. We also investigate the mean squared fields and the energy density in the vacuum region between two parallel half-spaces. Of particular interest is the sign of the energy density. We find that the energy density is described by two terms: a negative position independent (Casimir) term, and a positive position dependent term with a minimum value at the center of the vacuum region. We argue that in some cases, including physically realizable ones, the negative term can dominate in a given region between the two half-spaces, so the overall energy density can be negative in this region.

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1 Introduction

In 1948 Casimir made the remarkable prediction that there is an attractive force between a pair uncharged parallel plane perfect conductors [1]. Furthermore, he argued that this force arises solely from a shift in the energy of the vacuum state of the quantized electromagnetic field. An early attempt by Sparnaay [2] to observe this force was inconclusive, but in recent years several new experiments [3, 4, 5, 6, 7] have been performed which seem to give good agreement with Casimir's prediction. (To be more precise, most of these experiments actually measure the force between a plate and a sphere and incorporate a theoretical correction to compare to Casimir's result. Of the recent experiments, only that of Bressi *et al* [7] uses two parallel plates.)

If the energy of the vacuum state is zero in the limit of infinite plate separation, then the attractive force found by Casimir would seem to imply a negative vacuum energy at finite separation. In fact, Brown and Maclay [8] showed that for perfectly conducting plates, one has a constant negative vacuum energy density. This conclusion is of great theoretical interest, because negative energy density has the potential to cause some rather bizarre effects in gravity theory. (See, for example, Ref. [9] and references therein.) However, questions have been raised as to whether the negative energy density will still arise in a more realistic treatment in which the plates are not perfect conductors [10, 11]. In particular,

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Helfer and Lang [10] calculated the energy density outside of a single half-space filled with a nondispersive dielectric material and obtained a positive result. They interpreted this as a positive self-energy density associated with a single plate which would add to the negative interaction energy density between a pair of plates. Helfer and Lang conjecture that the net Casimir energy density might be positive when the self energy is accounted for. If this conjecture is correct, then the situation would be analogous to that of the energy density in classical electrostatics. A pair of oppositely charged particles have a negative interaction energy, but the net energy density, which is proportional to the square of the electric field, is always positive.

However, the Helfer and Lang calculation does not include dispersion, which is essential in a realistic treatment. Numerous authors, beginning with Lifshitz [12], have studied the effects of dispersion upon Casimir forces. However, these authors have been concerned with the force or the total energy, and not the local energy density. The purpose of this paper is to present a calculation of the Casimir energy density in a model in which dispersion is included. For this purpose, we will use the methods of source theory developed by Schwinger and coworkers [13, 14]. This is a method based upon the calculation of Green's functions which is especially well suited to dissipative materials, and was used by Schwinger *et al* [13] to rederive the results of Lifshitz. Milonni and Shih [15] have used conventional quantum electrodynamics to reproduce some of the results of source theory. There has also been considerable interest in recent years in quantization of the electromagnetic field inside dissipative materials using operator methods [16, 17, 18]. The relation between the results of the latter set of authors and those of Schwinger *et al* has not yet been clarified.

The outline of this paper is as follows: In Sect. 2 we review the source theory approach as applied to parallel interfaces of dielectric media. In Sect. 3 we compute the expectation values of the squares of the electric and magnetic fields in the vacuum region outside a half-space filled with a uniform dispersive dielectric. We extend this calculation to the case two parallel dielectric half-spaces and also discuss the energy density in Sect. 4. Conclusions are given in Sect. 5.

2 Green-Function Approach for Multilayer Dielectrics

This section is a review of the formalism of Schwinger *et al* [13]. One begins by writing the Maxwell equations for the macroscopic electromagnetic fields produced by an external polarization source $\mathbf{P}$, which formally describes the zero point fluctuations of the fields[3]

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A}, \\
\mathbf{E} &= -\dot{\mathbf{A}} - \nabla\phi, \\
\nabla \times \mathbf{B} &= \epsilon\dot{\mathbf{E}} + \dot{\mathbf{P}}, \\
\nabla \cdot (\epsilon\mathbf{E} + \mathbf{P}) &= 0,
\end{aligned}$$

where ϵ is the dielectric constant of the medium. The wave equation for the electric field resulting from the Maxwell equations is

[3]Heaviside-Lorentz units with $c = \hbar = 1$ will be used in this paper. Also, it is assumed that the magnetic permeability is unity.

$$-\nabla\times(\nabla\times\mathbf{E})-\epsilon\ddot{\mathbf{E}}=\ddot{\mathbf{P}}, \tag{1}$$

By assuming a linear relation between sources and fields, the electric field can be written as a spacetime integral

$$\mathbf{E}(x)=\int d^4x'\,\overleftrightarrow{\Gamma}(x,x')\,\mathbf{P}(x'), \tag{2}$$

where $x=(t,\mathbf{r})$, $x'=(t',\mathbf{r}')$ and $\overleftrightarrow{\Gamma}$ is a Green's dyadic, which satisfies (1) with a δ-function source. Let

$$\overleftrightarrow{\Gamma}(\mathbf{r},\mathbf{r}',\omega)=\int_{-\infty}^{\infty}d\tau\,e^{i\omega\tau}\,\overleftrightarrow{\Gamma}(x,x')\,, \tag{3}$$

where $\tau=t-t'$. From (1) and (2), it follows that $\overleftrightarrow{\Gamma}(\mathbf{r},\mathbf{r}',\omega)$ satisfies the following equation:

$$-\nabla\times(\nabla\times\overleftrightarrow{\Gamma})+\omega^2\epsilon\overleftrightarrow{\Gamma}=-\omega^2\,\overleftrightarrow{\mathbf{1}}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{4}$$

So far, the discussion has been purely classical. At this point, Schwinger *et al* [13] use source theory to identify the Green's dyadic $\overleftrightarrow{\Gamma}$ with an "effective product of electric fields"

$$i\hbar\langle E_j(\mathbf{r})\,E_k(\mathbf{r}')\rangle=\Gamma_{jk}(\mathbf{r},\mathbf{r}',\omega). \tag{5}$$

We can interpret this as the Fourier transform of the electric field correlation function. From the Maxwell equation $\nabla\times\mathbf{E}=-\dot{\mathbf{B}}$, one finds the corresponding expression for the magnetic field:

$$i\hbar\langle B_j(\mathbf{r})\,B_k(\mathbf{r}')\rangle=\epsilon_{jlm}\epsilon_{knp}(\nabla_l\nabla_n/\omega^2)\Gamma_{mp}(\mathbf{r},\mathbf{r}',\omega). \tag{6}$$

Note that $\hbar$ makes its first appearance in these expressions. These expressions can be identified with the vacuum expectation values of products of field operators, which appear in the more conventional field theory approach to quantization of the electromagnetic field. From now onward, we revert to units in which $\hbar=1$. In order to calculate the field correlation functions, one needs to find the Green's function Γ occurring in (4). This amounts to solving a classical boundary value problem.

The interfaces between the media are chosen to be perpendicular to the z direction, so for now it will only matter that the dielectric constant changes in the z direction only. Therefore, it is convenient to introduce a transverse spatial Fourier transform

$$\overleftrightarrow{\Gamma}(\mathbf{r},\mathbf{r}',\omega) = \int d\mathbf{k}_\perp\ 1(2\pi)^2 e^{i\mathbf{k}_\perp(\mathbf{r}-\mathbf{r}')_\perp}\overleftrightarrow{\Gamma}(z,z',\mathbf{k}_\perp,\omega), \tag{7}$$

where the vector $\mathbf{k}_\perp$ can be chosen to point along the $+x$ axis ($k = |\mathbf{k}_\perp|$).

Some components of $\overleftrightarrow{\Gamma}$ are found to be [13]

$$\begin{aligned}
\Gamma_{xx} &= -1\epsilon\delta(z-z') + 1\epsilon\partial\partial z 1\epsilon'\partial\partial z' g^B, && (8)\\
\Gamma_{yy} &= \omega^2 g^E, && (9)\\
\Gamma_{zz} &= -1\epsilon\delta(z-z') + k^2\epsilon\epsilon' g^B, && (10)\\
\Gamma_{xz} &= ik\epsilon\epsilon'\partial\partial z g^B, && (11)\\
\Gamma_{zx} &= -ik\epsilon\epsilon'\partial\partial z' g^B, && (12)
\end{aligned}$$

where $\epsilon' = \epsilon(z')$, and g^E, the "transverse electric", and g^B, the "transverse magnetic" Green's functions satisfy

$$\begin{aligned}
\left[-\partial^2\partial z^2 + k^2 - \omega^2\epsilon\right] g^E(z,z') &= \delta(z-z'), && (13)\\
\left[-\partial\partial z 1\epsilon\partial\partial z + k^2\epsilon - \omega^2\right] g^B(z,z') &= \delta(z-z'). && (14)
\end{aligned}$$

By introducing the quantity

$$\kappa^2 = k^2 - \omega^2\epsilon, \tag{15}$$

(2) can be written as:

$$\begin{aligned}
\left[-\partial^2\partial z^2 + \kappa^2\right] g^E(z,z^{'}) &= \delta(z-z'), && (16)\\
\left[-\partial\partial z 1\epsilon\partial\partial z + \kappa^2\epsilon\right] g^B(z,z') &= \delta(z-z'). && (17)
\end{aligned}$$

So, in order to find the field correlation functions as defined in (5) and (6) in a given situation, one needs to solve these equations with the appropriate boundary conditions. We consider here two cases.

3 One Interface Case

We now specialize the above discussion to a situation in which the inhomogeneity of the dielectric constant is due to a plane interface separating a dielectric substance from a vacuum:

$$\begin{aligned} z > 0 : \ \epsilon(z) &= 1, \\ z < 0 : \ \epsilon(z) &\equiv \epsilon_d. \end{aligned} \tag{18}$$

Here ϵ_d is a function of frequency, but not of position.

3.1 *Boundary Conditions*

In solving (16) and (17), we use the following boundary conditions. At $z = z'$, g is continuous but the derivative is discontinuous at this point [19]:

$$\partial g \partial z \Big|_{z\to z'_-}^{z\to z'_+} = -1. \tag{19}$$

At the boundary ($z = 0$) we use the conditions for continuity of E_x, E_y, ϵE_z, and B_i. The first three, as seen from (5), imply the continuity of Γ_{xx}, Γ_{yy}, and $\epsilon\Gamma_{zz}$ and subsequently, from (2), the continuity of g^E, g^B, and

$$1\epsilon\partial\partial z 1\epsilon'\partial\partial z' g^B.$$

The continuity of B_{xx} implies that of $\nabla_z\nabla_{z'}\Gamma_{yy}$, as seen from Eq. (31), which is given below. From this, using (5) and (9), we deduce the continuity of $\partial g^E/\partial z$.

The solutions g^E and g^B in the vacuum region have the form

$$g^E = e^{-\kappa_0|z-z'|} + re^{-\kappa_0(z+z')}2\kappa_0, \tag{20}$$
$$g^B = e^{-\kappa_0|z-z'|} + r'e^{-\kappa_0(z+z')}2\kappa_0, \tag{21}$$

where

$$r \equiv \kappa_0 - \kappa_1\kappa_0 + \kappa_1 \tag{22}$$
$$r' \equiv \kappa_0\epsilon_d - \kappa_1\kappa_0\epsilon_d + \kappa_1. \tag{23}$$

Here κ_0 and κ_1 represent the quantity κ as defined in (15) for the vacuum region ($\epsilon = 1$) , and for the dielectric half-space region ($\epsilon = \epsilon_d$) , respectively, and r and r' can be identified as reflection coefficients for two polarization states, $\perp$ and $\|$ respectively, corresponding to electric field vector being perpendicular or parallel to the plane of incidence of an linearly polarized electromagnetic wave [19].

3.2 *The Electric Field*

Using (5), we write the formal expectation value of the square of the electric field at coincident points as

$$\left\langle E^2\right\rangle_f = -i\int_{-\infty}^{\infty} d\omega 12\pi \int_0^{\infty} dk\, k\; 12\pi\Gamma_{kk} = -i2\pi^2 \int_0^{\infty} d\omega \int_0^{\infty} dk\, k\; \Gamma_{kk}. \tag{24}$$

In the second step, we assumed that the integrand is an even function of ω. By complex rotation ($\omega \to i\zeta$), this becomes:

$$\left\langle E^2\right\rangle_f = 12\pi^2 \int_0^{\infty} d\zeta \int_0^{\infty} dk\, k\; \Gamma_{kk}. \tag{25}$$

Note from (15) that $\kappa^2 > 0$ when ω is imaginary. By means of (2), all of the components of $\overleftrightarrow{\Gamma}$ in a given region can be written in terms of Γ_{xx} and Γ_{yy}:

$$\begin{array}{rcl} \Gamma_{xz}(z,z') & = & ik\kappa^2\partial\partial z'\,\Gamma_{xx}(z,z'), \\ \Gamma_{zx}(z,z') & = & -ik\kappa^2\partial\partial z\,\Gamma_{xx}(z,z'), \\ \Gamma_{zz}(z,z') & = & k^2(\kappa^2)\partial\partial z\partial\partial z'\,\Gamma_{xx}(z,z') + \omega^2\kappa^2\delta(z-z'). \end{array} \tag{26}$$

By taking the limit $z \to z'$, and thus omitting the delta function, Γ_{kk} becomes

$$\Gamma_{kk} = \Gamma_{xx} + \Gamma_{yy} + k^2(\kappa^2)^2\nabla_z\nabla_{z'}\Gamma_{xx}, \tag{27}$$

or by (2), using $\epsilon = 1$,

$$\begin{array}{rcl} \Gamma_{kk} & = & \omega^2 g^E + \nabla_z\nabla_{z'}g^B + k^2(\kappa^2)^2\nabla_z\nabla_{z'}(\nabla_z\nabla_{z'}g^B) \\ & = & \omega^2 g^E + (k^2 + \nabla_z\nabla_{z'})g^B. \end{array} \tag{28}$$

Using (15) and (3.1), this becomes

$$\Gamma_{kk} = \omega^2\kappa + 12\kappa\left[\omega^2 r + \left(2k^2 - \omega^2\right)r'\right]e^{-2\kappa z}. \tag{29}$$

Equation (25) gives a formal expectation value only because the integral is divergent. However, the divergence comes only from the ω^2/κ term in Γ_{kk} and is independent of z. It is the usual empty space vacuum divergence. We will henceforth drop this term and denote the resulting finite expectation vaule by $\left\langle E^2\right\rangle$. The renormalization results in a quantity which vanishes at large distances from the interface: $\left\langle E^2\right\rangle \to 0$ as $z \to \infty$, which amounts to finding the difference in $\left\langle E^2\right\rangle$ with the boundary and without it. Thus we find

$$\left\langle E^2\right\rangle = 14\pi^2 \int_0^\infty d\zeta \int_0^\infty dk\, k\kappa \left[-\zeta^2 r + \left(2k^2+\zeta^2\right) r'\right] e^{-2\kappa z}. \tag{30}$$

3.3 *The Magnetic Field*

Now we compute the expectation value of the magnetic field. Using Eq. (6), we find

$$\begin{aligned} i\langle B_x(\mathbf{r})\, B_x(\mathbf{r}')\rangle &= 1\omega^2 \left(\nabla_z\nabla_{z'}\Gamma_{yy} - \nabla_y\nabla_{z'}\Gamma_{zy} - \nabla_z\nabla_{y'}\Gamma_{yz} + \nabla_y\nabla_{y'}\Gamma_{zz}\right), \\ i\langle B_y(\mathbf{r}) B_y(\mathbf{r}')\rangle &= 1\omega^2 \left(\nabla_z\nabla_{z'}\Gamma_{xx} - \nabla_x\nabla_{z'}\Gamma_{zx} - \nabla_z\nabla_{x'}\Gamma_{xz} + \nabla_x\nabla_{x'}\Gamma_{zz}\right), \\ i\langle B_z(\mathbf{r})\, B_z(\mathbf{r}')\rangle &= 1\omega^2 \left(\nabla_y\nabla_{y'}\Gamma_{xx} - \nabla_y\nabla_{x'}\Gamma_{xy} - \nabla_x\nabla_{y'}\Gamma_{yx} + \nabla_x\nabla_{x'}\Gamma_{yy}\right). \end{aligned} \tag{31}$$

From the definition of $\mathbf{k}_\perp$, it follows that all derivatives in y vanish, so we can write the sum of the above terms as

$$\begin{aligned} i\langle B_i(\mathbf{r}) B_i(\mathbf{r}')\rangle = 1\omega^2 (&\nabla_z\nabla_{z'}\Gamma_{yy} + \nabla_z\nabla_{z'}\Gamma_{xx} - \\ &\nabla_x\nabla_{z'}\Gamma_{zx} - \nabla_z\nabla_{x'}\Gamma_{xz} + \nabla_x\nabla_{x'}\Gamma_{zz} + \nabla_x\nabla_{x'}\Gamma_{yy}). \end{aligned}$$

Using (26), we have

$$\begin{aligned} \nabla_x\nabla_{x'}\Gamma_{zz}(\mathbf{r},\mathbf{r}',\omega) &= \int d\mathbf{k}_\perp (2\pi)^2\, (k^2)^2(\kappa^2)^2\, \nabla_z\nabla_{z'}\Gamma_{xx}, \\ \nabla_z\nabla_{x'}\Gamma_{xz}(\mathbf{r},\mathbf{r}',\omega) &= \int d\mathbf{k}_\perp (2\pi)^2\, k^2\kappa^2\, \nabla_z\nabla_{z'}\Gamma_{xx}, \\ \nabla_x\nabla_{z'}\Gamma_{zx}(\mathbf{r},\mathbf{r}',\omega) &= \int d\mathbf{k}_\perp (2\pi)^2\, k^2\kappa^2\, \nabla_z\nabla_{z'}\Gamma_{xx}. \end{aligned} \tag{32}$$

This leads to

$$i\langle B(x)\, B(x')\rangle = \int d\omega 2\pi \int d\mathbf{k}_\perp (2\pi)^2 \left[1\omega^2 \left(k^2 + \nabla_z\nabla_{z'}\right)\Gamma_{yy}(z,z') + \omega^2\kappa^4\nabla_z\nabla_{z'}\Gamma_{xx}(z,z')\right]. \tag{33}$$

Using (2), this becomes

$$i\langle B(x)\,B(x')\rangle = \int d\omega 2\pi \int d\mathbf{k}_\perp (2\pi)^2 \left[\left(k^2 + \nabla_z \nabla_{z'}\right) g^E(z,z') + \omega^2 g^B(z,z')\right]. \tag{34}$$

Following the same procedure as used above in calculating $\langle E^2 \rangle$, we find the finite mean squared magnetic field to be

$$\langle B^2 \rangle = 14\pi^2 \int_0^\infty d\zeta \int_0^\infty dk\; k\kappa \left[\left(2k^2 + \zeta^2\right) r - \zeta^2 r'\right] e^{-2\kappa z}. \tag{35}$$

Note that the expressions (30) and (35) are symmetric under interchange of r and r'. Now, the mean energy density can be calculated as

$$U = 12 \left(\langle E^2 \rangle + \langle B^2 \rangle\right). \tag{36}$$

Using (30) and (35), this becomes

$$U = 14\pi^2 \int_0^\infty d\zeta \int_0^\infty dk\, k^3 \kappa \left(r + r'\right) e^{-2\kappa z}. \tag{37}$$

We can write U in a form more convenient for numerical calculation by introducing polar coordinates u and θ ($\zeta = u\cos\theta$, $k = u\sin\theta$):

$$U = 14\pi^2 \int_0^\infty du\, u^3 \int_0^{\pi 2} d\theta \,(\sin\theta)^3 \left(r + r'\right) e^{-2uz}. \tag{38}$$

We use the Drude model for the dielectric function

$$\epsilon_d(\omega) = 1 - \omega_p^2 \omega^2, \tag{39}$$

where ω_p is the plasma frequency. From (22), (23), and (39), we find

$$r = u - \sqrt{u^2 + \omega_p^2}\, u + \sqrt{u^2 + \omega_p^2}, \tag{40}$$

$$r' = u^2(\cos\theta)^2 + \omega_p^2 - u(\cos\theta)^2 \sqrt{u^2 + \omega_p^2}\, u^2(\cos\theta)^2 + \omega_p^2 + u(\cos\theta)^2 \sqrt{u^2 + \omega_p^2}. \tag{41}$$